# Ladder Theorem and length-scale estimates for a Leray alpha model of turbulence

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**Abstract**

In this paper, we study the Modified Leray alpha model with periodic boundary conditions. We show that the regular solution verifies a sequence of energy inequalities that is called "ladder inequalities". Furthermore, we estimate some quantities of physical relevance in terms of the Reynolds number.

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## 1 Introduction

We consider in this paper the ML-$\alpha$ model of turbulence

$$(1)\qquad \begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\overline{\boldsymbol{u}} - \nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} \qquad \text{in } \mathbb{R}^+ \times \mathbb{T}_3, \\ -\alpha^2\Delta\overline{\boldsymbol{u}} + \overline{\boldsymbol{u}} + \nabla\pi = \boldsymbol{u} \qquad \text{in } \mathbb{T}_3, \\ \nabla\cdot\boldsymbol{u} = \nabla\cdot\overline{\boldsymbol{u}} = 0, \\ \oint_{\mathbb{T}_3} \boldsymbol{u} = \oint_{\mathbb{T}_3} \overline{\boldsymbol{u}} = 0, \\ \boldsymbol{u}_{t=0} = \boldsymbol{u}^{in}. \end{cases}$$

The boundary conditions are periodic boundary conditions. Therefore we consider these equations on the three dimensional torus $\mathbb{T}_3 = \left(\mathbb{R}^3/\mathcal{T}_3\right)$ where $\mathcal{T}_3 = 2\pi\mathbb{Z}^3/L$ , $\mathbf{x} \in \mathbb{T}_3$, and $t \in ]0, +\infty[$. The unkowns are the velocity vector field $\boldsymbol{u}$ and the scalar pressure p. The viscosity $\nu$, the initial velocity vector field $\boldsymbol{u}^{in}$, and the external force $\boldsymbol{f}$ with $\nabla\cdot\boldsymbol{f} = 0$ are given. In this paper the force $\boldsymbol{f}$ does not depend on time.
This model has been first studied in [9], where the authors prove the global existence and uniqueness of the solution. They also prove the existence of a global attractor $\mathcal{A}$ to this model and they made estimates of the fractal dimension of this attractor in terms of Grashof number $Gr$.
The dimension of the attractor gives us some idea of the level of the complexity of the flow. The relation between the number of determining modes, determining nodes and the evolution of volume elements of the attractors are discussed by Jones and Titi in [13].

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Temam also interprets in his book [16] the dimension of the attractors as the number of degrees of freedom of the flow.
It is easily seen that when $\alpha = 0$, eqs. (1) reduce to the usual Navier Stokes equations for incompressible fluids.
Assuming that $\boldsymbol{f} \in C^\infty$, any $C^\infty$ solution to the Navier Stokes equations verifies formally what is called the ladder inequality [4]. That means, for any $C^\infty$ solution $(\boldsymbol{u}, p)$ to the (NSE), the velocity part $\boldsymbol{u}$ satisfies the following relation between its higher derivatives,

$$\text{(2)} \qquad \begin{aligned} &\frac{1}{2}\frac{d}{dt}H_N \leq -\nu H_{N+1} + C_N H_N \left\|\nabla \boldsymbol{u}\right\|_\infty + H_N^{1/2} {\Phi_N}^{1/2}, \\ &\text{where } H_N = \int_{\mathbb{T}_3} \left|\nabla^N \boldsymbol{u}\right|^2 d\mathbf{x} \text{ and } \Phi_N = \int_{\mathbb{T}_3} |\nabla^N \boldsymbol{f}|^2 d\mathbf{x}. \end{aligned}$$

This differential inequalities are used first in [4] to show the existence of a lower bound on the smallest scale in the flow. The same result is obtained in [5] by a Gevrey Class estimates. Recently, the ladder inequalities are used to study the intermittency of solutions to the Navier Stokes equations see [10]. While the ladder inequalities to the Navier Stokes equations are based on the assumption that a solution exists, so that the higher order norms are finite, no such assumption is necessary in the case of alpha regularistaion where existence and uniqueness of a $C^\infty$ solution are guaranteed. The ladder inequalties are generalized in [11, 12] to other equations based on the Navier stokes equations such as Navier Stokes-alpha model [6] and Leray alpha model [2].
We aim to study in this paper ladder inequalities for model (1). In the whole paper, $\alpha > 0$ is given and we assume that the initial data is $C^\infty$. One of the main results of this paper is:

**Theorem 1.1** *Assume $\boldsymbol{f} \in C^\infty(\mathbb{T}_3)^3$ and $\boldsymbol{u}^{in} \in C^\infty(\mathbb{T}_3)^3$. Let $(\boldsymbol{u}, p) := (\boldsymbol{u}^\alpha, p^\alpha)$ be the unique solution to problem (1).Then the velocity part $\boldsymbol{u}$ satisfies the ladder inequalities,*

$$\text{(3)} \qquad \begin{aligned} \frac{1}{2}\left(\frac{d}{dt}\overline{H_N} + \alpha^2 \frac{d}{dt}\overline{H_{N+1}}\right) \leq &-\nu(\overline{H_{N+1}} + \alpha^2 \overline{H_{N+2}}) \\ &+ C_N \left\|\nabla \overline{\boldsymbol{u}}\right\|_\infty (\overline{H_N} + \alpha^2 \overline{H_{N+1}}) + \overline{H_N}^{1/2} {\Phi_N}^{1/2}, \end{aligned}$$

*where*

$$\text{(4)} \qquad \overline{H_N} = \int_{\mathbb{T}_3} \left|\nabla^N \overline{\boldsymbol{u}}\right|^2 d\boldsymbol{x}, C_0 = 0 \text{ and } C_N \approx 2^N \text{ for all } N \geq 1.$$

The gradient symbol $\nabla^N$ here refers to all derivatives of evrey component of $\boldsymbol{u}$ of order $N$ in $L^2(\mathbb{T}_3)$.

**Remark 1.1** *We note that, as $\alpha \to 0$, $\overline{H_N} \to H_N$. Thus we find the inequality (2).*

Another Task of this paper is to estimate quantities of physical relevance in terms of the Reynolds number, these result are summarized in the table 1 whose proof is given in section 5. For simplicity the eqs. (1) will be considered with forcing $\boldsymbol{f}(\boldsymbol{x})$ taken to be $L^2$ bounded of narrow band type with a single lenght scale $\ell$ (see [10, 11]) such that

$$\text{(5)} \qquad \|\nabla^n \boldsymbol{f}\|_{L^2} \approx \ell^{-n} \|\boldsymbol{f}\|_{L^2}.$$

In order to estimate small length sacles associated with higher order moments, we combine in section 5 the force with the higher derivative of the velocity such that

$$\text{(6)} \qquad J_N = \overline{F_N} + 2\alpha^2 \overline{F_{N+1}},$$

where

$$\overline{F_N} = \overline{H_N} + \tau\Phi_N, \tag{7}$$

and the quantity $\tau$ is defined by

$$\tau = \ell^2\nu^{-1}(Gr \ln Gr)^{-1/2}. \tag{8}$$

The $J_N$ is used to define a set of time-depend inverse length scales

$$\kappa_{N,r} = \left(\frac{J_N}{J_r}\right)^{\frac{1}{2(N-r)}}. \tag{9}$$

The second main result of the paper is the following Theorem.

**Theorem 1.2** *Let $\boldsymbol{f} \in C^\infty(\mathbb{T}_3)^3$ with narrow-band type and $\boldsymbol{u}^{in} \in C^\infty(\mathbb{T}_3)^3$. Let $\boldsymbol{u} := \boldsymbol{u}^\alpha$ be the velocity part of the solution to problem (1). Then estimates in term of Reynolds number $Re$ for the length sacles associated with higher order moments solution $\kappa_{N,0}$ ($N \geq 2$), the inverse Kolomogrov length $\lambda_k$ and the attractor dimesion $d_{F,ML-\alpha}(\mathcal{A})$ are given by*

$$\ell^2 \left\langle \kappa_{N,0}^2 \right\rangle \leq C(\alpha, \nu, \ell, L)^{(N-1)/N} Re^{5/2-3/2N} (\ln Re)^{1/N} + C Re \ln Re. \tag{10}$$

$$\ell\lambda_k^{-1} \leq c Re^{5/8}. \tag{11}$$

$$d_{F,ML-\alpha}(\mathcal{A}) \leq c \left(\frac{L^3\ell^{-4}}{\alpha^2\lambda_1^{3/2}}\right)^{3/4} Re^{9/4}. \tag{12}$$

*Where $\langle\cdot\rangle$ is the long time average defined below (14)*

The paper is organized as follows: In section 2, we start by summarizing and discussing the results given above. In section 3 we recall some helpfuls results about existence and uniqueness for this ML$-\alpha$ model, and we prove a general regularity result. In section 4, we prove Theorem 1.1. We stress that for all $N \in \mathbb{N}$ fixed, inequality (3) goes to inequality (2) when $\alpha \to 0$, at least formally. In section 5, we prove Theorem 1.2.

# 2 Summary and discussion of the results

Generally the most important of the estimates in Navier stokes theory have been found in terms of the Grashof number $Gr$ defined below in terms of the forcing, but these are difficult to compare with the results of Kolomogrov scaling theories [8] which are expressed in terms of Reynolds number $Re$ based on the Navier Stokes velocity $\boldsymbol{u}$. A good definition of this is

$$Re = \frac{U\ell}{\nu}, \quad U^2 = L^{-3}\left\langle \|\boldsymbol{u}\|_{L^2}^2 \right\rangle, \tag{13}$$

where $\langle\cdot\rangle$ is the long time average

$$\langle g(\cdot)\rangle = \mathrm{Lim}_{t\to\infty}\frac{1}{t}\int_0^t g(s)ds. \tag{14}$$

Where Lim indicates a generalized limit that extends the usual limits [7].
With $f_{rms} = L^{-3/2}\|\boldsymbol{f}\|_{L^2}$, the standard definition of the Grashof number in three dimensions is

$$Gr = \frac{\ell^3 f_{rms}}{\nu^2}. \tag{15}$$

Doering and Foias [3] have addressed the problem of how to relate $Re$ and $Gr$ and have shown that in the limit $Gr \to \infty$, solutions of the Navier Stokes equations must satisfy

$$Gr \leq c(Re^2 + Re). \tag{16}$$

Using the above relation (16), Doering and Gibbon [10] have re-expressed some Navier Stokes estimates in terms of $Re$. In particular they showed that the energy dissipation rate $\epsilon = \nu \left\langle \|\nabla \boldsymbol{u}\|_{L^2}^2 \right\rangle L^{-3}$ is bounded above by

$$\epsilon \leq c\nu^3 \ell^{-4} \left(Re^3 + Re\right), \tag{17}$$

and the inverse kolomogrov length $\lambda_k^{-1} = (\epsilon/\nu^3)^{1/4}$ is bounded above by

$$\ell \lambda_k^{-1} \leq c Re^{3/4}. \tag{18}$$

The relation (16) is essentially a Navier Stokes result. In [11] it has been shown that this property holds for the Navier Stokes-alpha model [6]; the same methods can be used to show this also holds for eqs. (1). In this paper, we will use (16) to obtain estimates in terms of the Reynolds number $Re$.

| | NS | NS-$\alpha$/Bardina | Leray-$\alpha$ | ML-$\alpha$ | Eq. |
|---|---|---|---|---|---|
| $\ell\lambda_k^{-1}$ | $Re^{3/4}$ | $Re^{5/8}$ | $Re^{7/12}$ | $Re^{5/8}$ | (18) |
| $\langle \overline{H_1} \rangle$ | $Re^3$ | $Re^{5/2}$ | $Re^{7/3}$ | $Re^{5/2}$ | (72) |
| $\langle \overline{H_2} \rangle$ | - | $Re^3$ | $Re^{8/3}$ | $Re^3$ | (69) |
| $\langle \overline{H_3} \rangle$ | - | - / - | $Re^3$ | $Re^7$ | (60) |
| $d_F(\mathcal{A})$ | - | $Re^{9/4}$ / $Re^{9/5}$ | $Re^{9/7}$ | $Re^{9/4}$ | (75) |
| $\ell^2 \left\langle \kappa_{N,r}^2 \right\rangle$ | - | $Re^{11/4}$ | $Re^{17/4}$ | $Re^{5/2}$ | (62) |
| $\ell^2 \left\langle \kappa_{1,0}^2 \right\rangle$ | $Re \ln Re$ | $Re \ln Re$ | $Re \ln Re$ | $Re \ln Re$ | (57) |
| $\left\langle \|\overline{\boldsymbol{u}}\|_\infty^2 \right\rangle$ | - | $Re^{11/4}$ | $Re^{5/2}$ | $Re^{11/4}$ | (58) |
| $\langle \|\nabla \overline{\boldsymbol{u}}\|_\infty \rangle$ | - | $Re^{35/16}$ | $Re^{17/12}$ | $Re^{5/2}$ | (61) |
| $\ell^2 \left\langle \kappa_{N,0}^2 \right\rangle$ | - | $Re^{\frac{11}{4}-\frac{7}{4N}}(\ln Re)^{\frac{1}{N}}$ | $Re^{\frac{17}{12}-\frac{5}{12N}}(\ln Re)^{\frac{1}{N}}$ | $Re^{\frac{5}{2}-\frac{3}{2N}}(\ln Re)^{\frac{1}{N}}$ | (65) |

Table 1: Comparison of various upper bounds for the Navier Stokes, Navier Stokes-$\alpha$, Bardina, Leray-$\alpha$ and Modified Leray-$\alpha$ with constant omitted

These estimates are listed in Table 1. The estimate for $d_{F,ML-\alpha}(\mathcal{A})$ are consistent with the long-standing belief that $Re^{3/4} \times Re^{3/4} \times Re^{3/4}$ resolution grid points are needed to numerically resolve the flow. The fact that this bound is not valid to the Navier Stokes equations is consistent with the fact that $d_{F,ML-\alpha}(\mathcal{A})$ blows up as $\alpha$ tends to zero. The improved estimate to the inverse kolomgrov $\lambda_k^{-1}$ coincide with the estimate to the Navier Stokes alpha given in [11] and blows up when $\alpha$ tends to zero. The estimate for $\left\langle \kappa_{N,0}^2 \right\rangle$ comes out to be sharper than those given for the Navier Stokes alpha because of the $\|\nabla \boldsymbol{u}\|_\infty$ term in the ladder inequality as opposed to the $\nu^{-1}\|\boldsymbol{u}\|_\infty^2$ in [11]. This estimate gives us a length scale that is immensely small. Such scale is unreachable computationally and the regular solution on a neighbour of this scale is unresolvable. Thus the resolution issues in computations of the flow are not only associated with the problem of regularity but they also raise the question of how resolution length scales can be defined and estimated.

We finish this section by the following remark. The existence and the uniqueness of a $C^\infty$ solution for all time $T$ to the ML-$\alpha$ motivate the present study. Provided that regular solution exists for a maximal inteval time $[0,T^*[$, we can show the ladder inequalities to the Navier Stokes equations in $[0,T^*[$. We then naturally ask ourselves If can we use the convergence of (3) to (2) in $[0,T^*[$ to deduce some informations about the regular solution beyond the time $T^*$? This is an crucial problem.

# 3 Existence, unicity and Regularity results

We begin this section by recalling the system (1) considered with periodic boundary conditions.

$$
\left\{
\begin{array}{l}
\dfrac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\overline{\boldsymbol{u}} - \nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} \qquad \text{in } \mathbb{R}^+ \times \mathbb{T}_3, \\
-\alpha^2\Delta\overline{\boldsymbol{u}} + \overline{\boldsymbol{u}} + \nabla\pi = \boldsymbol{u} \qquad \text{in } \mathbb{T}_3, \\
\nabla\cdot\boldsymbol{u} = \nabla\cdot\overline{\boldsymbol{u}} = 0, \\
\displaystyle\oint_{\mathbb{T}_3} \boldsymbol{u} = \oint_{\mathbb{T}_3} \overline{\boldsymbol{u}} = 0, \\
\boldsymbol{u}_{t=0} = \boldsymbol{u}^{in}.
\end{array}
\right. \tag{19}
$$

Note that given $\boldsymbol{u} = \overline{\boldsymbol{u}} - \alpha^2\Delta\overline{\boldsymbol{u}}$ the Poincaré inequality $\|\boldsymbol{u}\|_{L^2} \le L/2\pi\|\nabla\boldsymbol{u}\|_{L^2}$ immediately leads to

$$
\alpha^2\|\overline{\boldsymbol{u}}\|_{H^2} \le \|\boldsymbol{u}\|_{L^2} \le (\frac{L^2}{4\pi^2} + \alpha^2)\|\overline{\boldsymbol{u}}\|_{H^2}. \tag{20}
$$

In order to proof the ladder inequalities (3) we need first to show a regularity result for (1) or (19).

**Proposition 3.1** *Assume $\boldsymbol{f} \in H^{m-1}(\mathbb{T}_3)^3$ and $\boldsymbol{u}^{in} \in H^m(\mathbb{T}_3)^3$, $m \ge 1$, then the solution $(\boldsymbol{u}, p)$ of (1) is such that*

$$
\boldsymbol{u} \in L^\infty([0,T], H^m(\mathbb{T}_3)^3) \cap L^2([0,T], H^{m+1}(\mathbb{T}_3)^3), \tag{21}
$$

$$
p \in L^2([0,T], H^m(\mathbb{T}_3)^3). \tag{22}
$$

The following Theorem is a direct consequence of proposition 3.1.

**Theorem 3.1** *Assume $\boldsymbol{f} \in C^\infty(\mathbb{T}_3)^3$ and $\boldsymbol{u}^{in} \in C^\infty(\mathbb{T}_3)^3$. Let $(\boldsymbol{u}, p)$ be the solution to problem (1). Then the solution is $C^\infty$ in space and time.*

The aim of this section is the proof of proposition 3.1. We begin by recalling some known result for (1) or (19).

## 3.1 Known results

Results in [9] can be summarised as follows:

**Theorem 3.2** *Assume* $\boldsymbol{f} \in L^2(\mathbb{T}_3)^3$ *and* $\boldsymbol{u}^{in} \in H^1(\mathbb{T}_3)^3$. *Then for any* $T > 0$, *(1) has a unique distributional solution* $(\boldsymbol{u}, p) := (\boldsymbol{u}^\alpha, p^\alpha)$ *such that*

$$\boldsymbol{u} \in L^\infty([0,T], H^{-1}(\mathbb{T}_3)^3) \cap L^2([0,T], L^2(\mathbb{T}_3)^3), \tag{23}$$

$$\overline{\boldsymbol{u}} \in L^\infty([0,T], H^1(\mathbb{T}_3)^3) \cap L^2([0,T], H^2(\mathbb{T}_3)^3), \tag{24}$$

$$\begin{aligned} \|\overline{\boldsymbol{u}}(t)\|_{L^2}^2 + \alpha^2\|\overline{\boldsymbol{u}}(t)\|_{H^1}^2 \leq (\|\boldsymbol{u}^{in}\|_{L^2}^2 + \alpha^2\|\boldsymbol{u}^{in}\|_{H^1}^2) \exp\left(-4\pi\nu t/L^2\right) \\ + \frac{L^2}{4\pi^2\nu^2}\|\boldsymbol{f}\|_{H^{-1}}^2 \left(1 - \exp\left(-4\pi\nu t/L^2\right)\right). \end{aligned} \tag{25}$$

*Furthermore, if* $\boldsymbol{u}^{in} \in H^2(\mathbb{T}_3)^3$ *then*

$$\boldsymbol{u} \in L^\infty([0,T], L^2(\mathbb{T}_3)^3), \tag{26}$$

$$\overline{\boldsymbol{u}} \in L^\infty([0,T], H^2(\mathbb{T}_3)^3), \tag{27}$$

$$\|\overline{\boldsymbol{u}}(t)\|_{H^1}^2 + \alpha^2\|\overline{\boldsymbol{u}}(t)\|_{H^2}^2 \leq k(t). \tag{28}$$

*Where* $k(t)$ *verifies in particular:*
*(i)* $k(t)$ *is finite for all* $t > 0$.
*(ii)* $\limsup_{t\to\infty} k(t) < \infty$.

**Remark 3.1** *(1) The proof is based on the following energy inequality that is obtained by taking the inner product of (1) with* $\overline{\boldsymbol{u}}$,

$$\frac{1}{2}\left(\frac{d}{dt}\|\overline{\boldsymbol{u}}\|_{L^2}^2 + \alpha^2\frac{d}{dt}\|\nabla\overline{\boldsymbol{u}}\|_{L^2}^2\right) + \nu(\|\nabla\overline{\boldsymbol{u}}\|_{L^2}^2 + \alpha^2\|\Delta\overline{\boldsymbol{u}}\|_{L^2}^2) \leq \|\boldsymbol{f}\|_{L^2}\|\overline{\boldsymbol{u}}\|_{L^2}. \tag{29}$$

*(2) Note that the pressure may be reconstructed from* $\boldsymbol{u}$ *and* $\overline{\boldsymbol{u}}$ *by solving the elliptic equation*

$$\Delta p = \nabla \cdot ((\boldsymbol{u} \cdot \nabla)\overline{\boldsymbol{u}}).$$

*One concludes from the classical elliptic theory that* $p \in L^1([0,T], L^2(\mathbb{T}_3)^3)$.

We recall that we can extract subsequences of solution that converge as $\alpha \to 0$ to a weak solution of the Navier Stokes equations. The reader can look in [9], [6] and [1] for more details.

**Corollary 3.1** *(1) We have* $\boldsymbol{u} \in L^2([0,T], L^2(\mathbb{T}_3))$ *and by Sobolev embending* $\overline{\boldsymbol{u}} \in L^2([0,T], L^\infty(\mathbb{T}_3))$. *Thus there exists a constant* $M(T) := M(\boldsymbol{u}^{in}, \boldsymbol{f}, \alpha, T) > 0$ *such that*

$$\int_0^t \|\overline{\boldsymbol{u}}\|_{L^\infty}^2 \leq \frac{1}{\alpha^2}\int_0^t \|\boldsymbol{u}\|_{L^2}^2 \leq M(T) \quad \textit{for all } t \in [0,T].$$

*(2) We also observe by using (20) that there exists a constant* $C(\alpha) := C(\alpha, L) > 0$ *such that*

$$\|\boldsymbol{u}(t)\|_{L^2}^2 \leq C(\alpha)k(t) \textit{ for all } t > 0. \tag{30}$$

### 3.2 Regularity: Proof of proposition 3.1

The proof of proposition 3.1 is classical (see for example in [14]). In order to make the paper self-contained we will give a complete proof for this regularity result. The proof is given in many steps.
**Step 1:** we show that $\boldsymbol{u} \in L^\infty([0,T], L^2(\mathbb{T}_3)^3) \cap L^2([0,T], H^1(\mathbb{T}_3)^3)$.
**Step 2:** we take $\partial_t \boldsymbol{u}$ as a test function in (1).
**Step 3:** We take the $m-1$ derivative of (1) then we take $\partial_t \nabla^{m-1}\boldsymbol{u}$ as a test function and the result follows by induction.

**Step 1:**
We have the following Lemma.

**Lemma 3.1** *For $\boldsymbol{u}^{in} \in L^2(\mathbb{T}_3)^3$ and $\boldsymbol{f} \in H^{-1}(\mathbb{T}_3)^3$, eqs. (1) have a unique solution $(\boldsymbol{u}, p)$ such that*

$$\boldsymbol{u} \in L^\infty([0,T], L^2(\mathbb{T}_3)^3) \cap L^2([0,T], H^1(\mathbb{T}_3)^3). \tag{31}$$

**Proof of Lemma 3.1.** We show formal a priori estimates for the solution established in Theorem 3.2. These estimates can be obtained rigorously using the Galerkin procedure. We take the inner product of (1) with $\boldsymbol{u}$ to obtain

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}(t,\mathbf{x})\|_{L^2}^2 + \nu\|\nabla \boldsymbol{u}(t,\mathbf{x})\|_{L^2}^2 \le \|\nabla^{-1}\boldsymbol{f}\|_{L^2}\|\nabla \boldsymbol{u}\|_{L^2} + |((\boldsymbol{u}\cdot\nabla)\overline{\boldsymbol{u}}, \boldsymbol{u})|. \tag{32}$$

Integration by parts and Cauchy-Schwarz inequality yield to

$$|((\boldsymbol{u}\cdot\nabla)\overline{\boldsymbol{u}}, \boldsymbol{u})| \le \|\boldsymbol{u}\otimes\overline{\boldsymbol{u}}\|_{L^2}\|\nabla \boldsymbol{u}\|_{L^2} \tag{33}$$

and by Young's inequality, we obtain

$$\begin{aligned} \|\nabla^{-1}\boldsymbol{f}\|_{L^2}\|\nabla \boldsymbol{u}\|_{L^2} &\le \frac{1}{\nu}\|\nabla^{-1}\boldsymbol{f}\|_{L^2}^2 + \frac{\nu}{4}\|\nabla \boldsymbol{u}\|_{L^2}^2, \\ |((\boldsymbol{u}\cdot\nabla)\overline{\boldsymbol{u}}, \boldsymbol{u})| &\le \frac{1}{\nu}\|\boldsymbol{u}\otimes\overline{\boldsymbol{u}}\|_{L^2}^2 + \frac{\nu}{4}\|\nabla \boldsymbol{u}\|_{L^2}^2. \end{aligned} \tag{34}$$

From the above inequalities we get

$$\begin{aligned} \frac{d}{dt}\|\boldsymbol{u}(t,\mathbf{x})\|_{L^2}^2 + \nu\|\nabla \boldsymbol{u}(t,\mathbf{x})\|_{L^2}^2 &\le \frac{2}{\nu}\|\nabla^{-1}\boldsymbol{f}\|_{L^2}^2 + \frac{2}{\nu}\|\boldsymbol{u}\overline{\boldsymbol{u}}\|_{L^2}^2 \\ &\le \frac{2}{\nu}\|\nabla^{-1}\boldsymbol{f}\|_{L^2}^2 + \frac{2}{\nu}\frac{1}{\alpha^2}\|\boldsymbol{u}\|_{L^2}^4, \end{aligned} \tag{35}$$

where we have used in the last step that

$$\|\overline{\boldsymbol{u}}\|_{L^\infty}^2 \le \frac{1}{\alpha^2}\|\boldsymbol{u}\|_{L^2}^2. \tag{36}$$

This implies that

$$\frac{d}{dt}(1 + \|\boldsymbol{u}(t,\mathbf{x})\|_{L^2}^2) \le C_1(1 + \|\boldsymbol{u}(t,\mathbf{x})\|_{L^2}^2)^2, \tag{37}$$

where $C_1 = \max(\frac{2}{\nu}\frac{1}{\alpha^2}, \dfrac{2}{\nu}\|\nabla^{-1}\boldsymbol{f}\|_{L^2}^2)$, and by Gronwall's Lemma, since $\|\boldsymbol{u}\|_{L^2}^2 \in L^1([0,T])$ (Corollary 3.1) we conclude that

$$1 + \|\boldsymbol{u}(t,\mathbf{x})\|_{L^\infty([0,T],L^2)}^2 \le K_1(T),$$

where $K_1(T) := K_1(T, \boldsymbol{u}^{in}, \boldsymbol{f})$ is given by

$$K_1(T) = (1 + \|\boldsymbol{u}^{in}\|_{L^2}^2) \exp\left( C_1 \int_0^T (1 + \|\boldsymbol{u}(s)\|_{L^2}^2) ds \right).$$

Furthermore, for every $T > 0$ we have from (35),

$$\|\boldsymbol{u}(T, \mathbf{x})\|_{L^2}^2 + \nu \int_0^T \|\nabla \boldsymbol{u}(t, \mathbf{x})\|_{L^2}^2 dt \leq \|\boldsymbol{u}^{in}\|_{L^2}^2 + \frac{2}{\nu}\|\nabla^{-1} \boldsymbol{f}\|_{L^2}^2 T + \frac{2}{\nu} K_1 M. \tag{38}$$

Thus $\boldsymbol{u} \in L^2([0, T], H^1(\mathbb{T}_3)^3)$ for all $T > 0$.

**Step 2:**
With the same assumption in the inital data as in Theorem 3.2 , we can find the following result:

**Lemma 3.2** *Assume $\boldsymbol{f} \in L^2(\mathbb{T}_3)^3$ and $\boldsymbol{u}^{in} \in H^1(\mathbb{T}_3)^3$ . Then for any $T > 0$, eqs. (1) have a unique regular solution $(\boldsymbol{u}, p)$ such that*

$$\boldsymbol{u} \in C([0, T], H^1(\mathbb{T}_3)^3) \cap L^2([0, T], H^2(\mathbb{T}_3)^3), \tag{39}$$

$$\frac{d\boldsymbol{u}}{dt} \in L^2([0, T], L^2(\mathbb{T}_3)^3), \tag{40}$$

$$p \in L^2([0, T], H^1(\mathbb{T}_3)^3). \tag{41}$$

**Proof of Lemma 3.2** It is easily checked that since $\boldsymbol{u} \in L^\infty([0, T], L^2(\mathbb{T}_3)^3) \cap L^2([0, T], H^1(\mathbb{T}_3)^3)$, then $\overline{\boldsymbol{u}} \in L^\infty([0, T], H^2(\mathbb{T}_3)^3) \cap L^2([0, T], H^3(\mathbb{T}_3)^3)$. Consequently, by Sobolev injection Theorem, we deduce that $\overline{\boldsymbol{u}} \in L^\infty([0, T], L^\infty(\mathbb{T}_3)^3)$ and $\nabla\overline{\boldsymbol{u}} \in L^2([0, T], L^\infty(\mathbb{T}_3)^3)$. Therefore,

$$(\boldsymbol{u} \cdot \nabla)\overline{\boldsymbol{u}} \in L^2([0, T], L^2(\mathbb{T}_3)^3). \tag{42}$$

Now, for fixed t, we can take $\partial_t \boldsymbol{u}$ as a test function in (1) and the procedure is the same as the one in [15]. Note that the proof given in [15] is formal and can be obtained rigorously by using Galerkin method combined with (42).
Once we obtain that $\boldsymbol{u} \in L^\infty([0, T], H^1(\mathbb{T}_3)^3) \cap L^2([0, T], H^2(\mathbb{T}_3)^3) \cap H^1([0, T], L^2(\mathbb{T}_3)^3)$ and $p \in L^2([0, T], H^1(\mathbb{T}_3)^3)$. Interpolating between $L^2([0, T], H^2(\mathbb{T}_3)^3)$ and $H^1([0, T], L^2(\mathbb{T}_3)^3)$ yields to $\boldsymbol{u} \in C([0, T], H^1(\mathbb{T}_3)^3)$.

**Step 3:**
We proceed by induction. The case $m = 1$ follows from Lemma 3.2.
Assume that for any $k = 1, ..., m-1$, if $\boldsymbol{f} \in H^{k-1}(\mathbb{T}_3)^3$ and $\boldsymbol{u}^{in} \in H^k(\mathbb{T}_3)^3$ then $\boldsymbol{u} \in L^\infty([0, T], H^k(\mathbb{T}_3)^3) \cap L^2([0, T], H^{k+1}(\mathbb{T}_3)^3)$ holds.
It remains to prove that when $k = m$, $\boldsymbol{f} \in H^{m-1}(\mathbb{T}_3)^3$ and $\boldsymbol{u}^{in} \in H^m(\mathbb{T}_3)^3$ that $\boldsymbol{u} \in L^\infty([0, T], H^m(\mathbb{T}_3)^3) \cap L^2([0, T], H^{m+1}(\mathbb{T}_3)^3)$.
It is easily checked that for $\boldsymbol{u} \in L^\infty([0, T], H^k(\mathbb{T}_3)^3) \cap L^2([0, T], H^{k+1}(\mathbb{T}_3)^3)$,
$\overline{\boldsymbol{u}} \in L^\infty([0, T], H^{k+2}(\mathbb{T}_3)^3) \cap L^2([0, T], H^{k+3}(\mathbb{T}_3)^3)$. Consequently, by Sobolev injection Theorem, we deduce that $\nabla^k\overline{\boldsymbol{u}} \in L^\infty([0, T], L^\infty(\mathbb{T}_3)^3)$, and $\nabla^{k+1}\overline{\boldsymbol{u}} \in L^2([0, T], L^\infty(\mathbb{T}_3)^3)$.
By taking the $m-1$ derivative of (1) we get in the sense of the distributions that

$$\begin{cases} \dfrac{\partial \nabla^{m-1}\boldsymbol{u}}{\partial t} + \nabla^{m-1}\left((\boldsymbol{u} \cdot \nabla)\overline{\boldsymbol{u}}\right) - \nu\nabla^{m-1}\Delta\boldsymbol{u} + \nabla^{m-1}\nabla p = \nabla^{m-1}\boldsymbol{f}, \\ \nabla \cdot \nabla^{m-1}\boldsymbol{u} = 0, \\ \nabla^{m-1}\boldsymbol{u}_{t=0} = \nabla^{m-1}\boldsymbol{u}^{in}. \end{cases} \tag{43}$$

where boundary conditions remain periodic and still with zero mean and the initial condition with zero divergence and mean.
Therefore, after using Leibniz Formula

$$\nabla^{m-1}\left((\boldsymbol{u}\cdot\nabla)\overline{\boldsymbol{u}}\right)=\sum_{k=0}^{m-1}C_{m-1}^{k}\nabla^{k}\boldsymbol{u}\nabla^{m-k}\overline{\boldsymbol{u}}, \tag{44}$$

since

$$\nabla^{k}\boldsymbol{u}\in L^{\infty}([0,T],L^{2}(\mathbb{T}_3)^3)$$

and

$$\nabla^{k+1}\overline{\boldsymbol{u}}\in L^{2}([0,T],L^{\infty}(\mathbb{T}_3)^3),$$

for any $k=1,...,m-1$.
We deduce that

$$\nabla^{m-1}\left((\boldsymbol{u}\cdot\nabla)\overline{\boldsymbol{u}}\right)\in L^{2}([0,T],L^{2}(\mathbb{T}_3)^3). \tag{45}$$

Now, for fixed t, we can take $\partial_t\nabla^{m-1}\boldsymbol{u}$ as a test function in (43) and the procedure is the same as the one in [15]. One obtains that $\boldsymbol{u}\in L^{\infty}([0,T],H^{m}(\mathbb{T}_3)^3)\cap L^{2}([0,T],H^{m+1}(\mathbb{T}_3)^3)$ and $p\in L^{2}([0,T],H^{m}(\mathbb{T}_3)^3)$. This finishes the proof of proposition 3.1.

# 4 Ladder Inequalities: Proof of theorem 1.1.

The first step in the proof of theorem 1.1, which has been expressed in section 1, is the energy inequality (29) that corresponding to the case $N=0$ of (3). Having showing in the above section the regularity result for (1). We can take the $N$ derivative of (1), we get in the sense of the distributions that for all $N\geq 1$,

$$\begin{cases} \dfrac{\partial\nabla^{N}\boldsymbol{u}}{\partial t}+\nabla^{N}\left((\boldsymbol{u}\cdot\nabla)\overline{\boldsymbol{u}}\right)-\nu\nabla^{N}\Delta\boldsymbol{u}+\nabla^{N}\nabla p=\nabla^{N}\boldsymbol{f}, \\ \nabla\cdot\nabla^{N}\boldsymbol{u}=0, \\ \nabla^{N}\boldsymbol{u}_{t=0}=\nabla^{N}\boldsymbol{u}^{in}. \end{cases} \tag{46}$$

where boundary conditions remain periodic and still with zero mean and the initial condition with zero divergence and mean. Taking $\nabla^{N}\boldsymbol{u}$ as test function in (46), we can write that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}_3}\left|\nabla^{N}\overline{\boldsymbol{u}}\right|^2 d\mathbf{x}+\alpha^2\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}_3}\left|\nabla^{N+1}\overline{\boldsymbol{u}}\right|^2 d\mathbf{x}=\nu\int_{\mathbb{T}_3}\nabla^{N}\overline{\boldsymbol{u}}\nabla^{N}\Delta\overline{\boldsymbol{u}}d\mathbf{x}-\nu\alpha^2\int_{\mathbb{T}_3}\nabla^{N}\overline{\boldsymbol{u}}\nabla^{N}\Delta\Delta\overline{\boldsymbol{u}}d\mathbf{x}$$
$$+\int_{\mathbb{T}_3}\nabla^{N}\overline{\boldsymbol{u}}\nabla^{N}((\overline{\boldsymbol{u}}\cdot\nabla)\overline{\boldsymbol{u}})d\mathbf{x}-\alpha^2\int_{\mathbb{T}_3}\nabla^{N}\overline{\boldsymbol{u}}\nabla^{N}((\Delta\overline{\boldsymbol{u}}\cdot\nabla)\overline{\boldsymbol{u}})d\mathbf{x}+\int_{\mathbb{T}_3}\nabla^{N}\overline{\boldsymbol{u}}\nabla^{N}\boldsymbol{f}d\mathbf{x}.$$

Where the pressure term vanishes as $\nabla\cdot\nabla^{N}\boldsymbol{u}=0$.
Using the defintion of $\overline{H_N}$ in (4) we obtain
(47)

$$\frac{1}{2}\left(\frac{d}{dt}\overline{H_N}+\alpha^2\frac{d}{dt}\overline{H_{N+1}}\right)\leq-\nu\left(\overline{H_{N+1}}+\alpha^2\overline{H_{N+2}}\right)+\left|\int_{\mathbb{T}_3}\nabla^{N}\overline{\boldsymbol{u}}\nabla^{N}((\overline{\boldsymbol{u}}\cdot\nabla)\overline{\boldsymbol{u}})d\mathbf{x}\right|$$
$$+\alpha^2\left|\int_{\mathbb{T}_3}\nabla^{N+1}\overline{\boldsymbol{u}}\nabla^{N-1}((\Delta\overline{\boldsymbol{u}}\cdot\nabla)\overline{\boldsymbol{u}})d\mathbf{x}\right|+\left|\int_{\mathbb{T}_3}\nabla^{N}\overline{\boldsymbol{u}}\nabla^{N}\boldsymbol{f}d\mathbf{x}\right|.$$

Where we have integrated by parts in the Laplacien terms.
The central terms are

$$\mathrm{NL}_1 = |\int_{\mathbb{T}_3} \nabla^N \overline{\boldsymbol{u}} \nabla^N ((\overline{\boldsymbol{u}} \cdot \nabla)\overline{\boldsymbol{u}}) d\mathbf{x}| \tag{48}$$

and

$$\mathrm{NL}_2 = \alpha^2 |\int_{\mathbb{T}_3} \nabla^{N+1} \overline{\boldsymbol{u}} \nabla^{N-1} ((\Delta\overline{\boldsymbol{u}} \cdot \nabla)\overline{\boldsymbol{u}}) d\mathbf{x}| \tag{49}$$

These two terms $\mathrm{NL}_1$ and $\mathrm{NL}_2$ can be bounded using the following Gagliardo-Nirenberg interpolation inequality [4]:

**Lemma 4.1** *The Gagliardo-Nirenberg interpolation inequality is:*
*For $1 \leq q, r \leq \infty$, $j$ and $m$ such that $0 \leq j < m$ we have*

$$\left\|\nabla^j \boldsymbol{v}\right\|_p \leq C \left\|\nabla^m \boldsymbol{v}\right\|_r^a \left\|\boldsymbol{v}\right\|_q^{1-a} \tag{50}$$

*where*

$$\frac{1}{p} = \frac{j}{d} + a\left(\frac{1}{r} - \frac{m}{d}\right) + \frac{1-a}{q}$$

*for $\frac{j}{m} \leq a < 1$ and $a = \frac{j}{m}$ if $m - j - \frac{d}{r} \in N^*$ .*

The first nonlinear term $\mathrm{NL}_1$ is estimated with the Gagliardo-Nirenberg inequality [4] by $c_N \left\|\nabla\overline{\boldsymbol{u}}\right\|_\infty \overline{H_N}$, where $c_0 = 0$ and $c_N \leq c2^N$. Indeed, the nonlinear first term $\mathrm{NL}_1$ is found to satisfy

$$\mathrm{NL}_1 = \left|\int_{\mathbb{T}_3} \nabla^N \overline{\boldsymbol{u}} \nabla^N ((\overline{\boldsymbol{u}} \cdot \nabla)\overline{\boldsymbol{u}}) d\mathbf{x}\right| \leq 2^N \overline{H_N}^{1/2} \sum_{l=1}^{N} \left\|\nabla^l \overline{\boldsymbol{u}}\right\|_{L^p} \left\|\nabla^{N+1-l} \overline{\boldsymbol{u}}\right\|_{L^q},$$

where $p$ and $q$ satisfy $1/p + 1/q = 1/2$ according to the Hölder inequality. We use now the two Gagliardo-Nirenberg inequalities

$$\left\|\nabla^l \overline{\boldsymbol{u}}\right\|_{L^p} \leq c_1 \left\|\nabla^N \overline{\boldsymbol{u}}\right\|_{L^2}^a \left\|\nabla\overline{\boldsymbol{u}}\right\|_\infty^{1-a},$$

$$\left\|\nabla^{N+1-l} \overline{\boldsymbol{u}}\right\|_{L^q} \leq c_2 \left\|\nabla^N \overline{\boldsymbol{u}}\right\|_{L^2}^b \left\|\nabla\overline{\boldsymbol{u}}\right\|_\infty^{1-b}.$$

Where $a$ anb $b$ must satisfy

$$\frac{1}{p} = \frac{l-1}{3} + a\left(\frac{1}{2} - \frac{N-1}{3}\right),$$

$$\frac{1}{q} = \frac{N-l}{3} + b\left(\frac{1}{2} - \frac{N-1}{3}\right).$$

Since $1/p + 1/q = 1/2$, we deduce $a + b = 1$. Thus we obtain

$$\left|\int_{\mathbb{T}_3} \nabla^N \overline{\boldsymbol{u}} \nabla^N ((\overline{\boldsymbol{u}} \cdot \nabla)\overline{\boldsymbol{u}}) d\mathbf{x}\right| \leq c_N \left\|\nabla\overline{\boldsymbol{u}}\right\|_\infty \overline{H_N}. \tag{51}$$

In the same way, we can estimate the nonlinear second term with Gagliardo-Nirenberg inequality in order to have

$$\alpha^2 |\int_{\mathbb{T}_3} \nabla^{N+1} \overline{\boldsymbol{u}} \nabla^{N-1} ((\Delta\overline{\boldsymbol{u}} \cdot \nabla)\overline{\boldsymbol{u}}) d\mathbf{x}| \leq c_N^{'} \alpha^2 \left\|\nabla\overline{\boldsymbol{u}}\right\|_\infty \overline{H_{N+1}}, \tag{52}$$

where $c_N^{'} \leq c2^N$.
The result (3) then follows.

# 5 Estimates in terms of Reynolds number: Proof of Theorem 1.2

## 5.1 Proof of inequality (10)

We begin by forming the combination

$$\overline{F_N} = \overline{H_N} + \tau \Phi_N$$

where the quantity $\tau$ is defined by

$$\tau = \ell^2 \nu^{-1} (Gr \ln Gr)^{-1/2}.$$

We define the combination

$$J_N = \overline{F_N} + 2\alpha^2 \overline{F_{N+1}}$$

The following result is a consequence of Theorem 1.1 and it is proof follows closely to that for the Navier Stokes-alpha model in [11] and we will not repeat it.

**Theorem 5.1** *As $Gr \to \infty$, for $N \geq 1$, $1 \leq p \leq N$ the unique solution to eqs. (1) satisfies*

$$\frac{1}{2}\frac{d}{dt} J_N \leq -\nu \frac{J_N^{1+\frac{1}{p}}}{J_{N-p}^{\frac{1}{p}}} + C_{N,\alpha} \|\nabla \overline{\boldsymbol{u}}\|_\infty J_N + C\nu\ell^{-2} Re(\ln Re) J_N \tag{53}$$

*and for $N = 0$,*

$$\frac{1}{2}\frac{d}{dt} J_0 \leq -\nu J_1 + C\nu\ell^{-2} Re(\ln Re) J_0 \tag{54}$$

When $\alpha \to 0$, $J_N$ tends to $F_N = H_N + \tau \Phi_N$, and the result of theorem 5.1 is consistent with the result achived to Navier Stokes equations in [4].
To obtain length scales estimates let us define the quantities

$$\kappa_{N,r} = \left( \frac{J_N}{J_r} \right)^{\frac{1}{2(N-r)}}$$

In the $\alpha \to 0$ limit, the $\kappa_{N,0}$ behaves as the $2N$th moment of the energy spectrum.
The aim of this subsection is to find a estimate for the length sacles associated with higher order moments solution $\kappa_{N,0}$ ($N \geq 2$). To this end, we find first upper bounds for $\left\langle \kappa_{N,r}^2 \right\rangle$, $\left\langle \kappa_{1,0}^2 \right\rangle$ and $\langle \|\nabla \overline{\boldsymbol{u}}\|_\infty \rangle$ . Then we use the following identity

$$\kappa_{N,0}^2 = \kappa_{N,1}^{2(N-1)/N} \kappa_{1,0}^{2/N} \tag{55}$$

in order to deduce the result.
(a) The First two bounds are obtained by dividing by $J_N$ in Theorem 5.1 and time averaging to obtain

$$\left\langle \kappa_{N,r}^2 \right\rangle \leq C_{N,\alpha} \nu^{-1} \langle \|\nabla \overline{\boldsymbol{u}}\|_\infty \rangle + C\ell^{-2} Re(\ln Re) \tag{56}$$

and

$$\left\langle \kappa_{1,0}^2 \right\rangle \leq C\ell^{-2} Re(\ln Re). \tag{57}$$

**Remark 5.1** *Note that the bound on* $\left\langle \|\overline{\boldsymbol{u}}\|_\infty^2 \right\rangle$ *is found to satisfies (see in [11] for more details),*

$$\left\langle \|\overline{\boldsymbol{u}}\|_\infty^2 \right\rangle \leq C\ell^{-2}\nu^2 V_\alpha Re^{11/4} \tag{58}$$

*Where*

$$V_\alpha := \left(\frac{L}{(\ell\alpha)^{1/2}}^3\right).$$

(b) It is also possible to estimate $\langle \|\nabla\overline{\boldsymbol{u}}\|_\infty \rangle$: return to the eqs. (1) and take a different way. We take $\boldsymbol{u} = -\alpha^2\Delta\overline{\boldsymbol{u}} + \overline{\boldsymbol{u}}$ as test function, then integration by parts (see Lemma 3.1), using (28) and time averaging, we obtain

(59)

$$\begin{aligned}
\nu\left\langle \overline{H_1} + 2\alpha^2\overline{H_2} + \alpha^4\overline{H_3} \right\rangle &\leq C\left\langle \|\nabla\overline{\boldsymbol{u}}\|_{L^2}\|\Delta\overline{\boldsymbol{u}}\|_{L^2}^2 \right\rangle + (1+\alpha^2\ell^{-2})\left\langle \overline{H_0}^{1/2}\Phi_0{}^{1/2} \right\rangle \\
&\leq C\left\langle \|\Delta\overline{\boldsymbol{u}}\|_{L^2}^2 \right\rangle \|\overline{\boldsymbol{u}}\|_{L^\infty([0,T],H^1)} + (1+\alpha^2\ell^{-2})\left\langle \overline{H_0}^{1/2}\Phi_0{}^{1/2} \right\rangle \\
&\leq C\alpha^{-2}\nu^2\ell^{-4}L^3Re^3Gr^2 + C(1+\alpha^2\ell^{-2})\nu^3\ell^{-4}L^3Re^3.
\end{aligned}$$

Thus we can right

$$\left\langle \overline{H_3} \right\rangle \leq C(\alpha,\nu,\ell,L)Re^7. \tag{60}$$

This can be used to find the estimate for $\langle \|\nabla\overline{\boldsymbol{u}}\|_\infty \rangle$. In fact, Agmon's inequality [7]

$$\|\boldsymbol{u}\|_\infty \leq \|\boldsymbol{u}\|_{H^1}^{1/2}\|\boldsymbol{u}\|_{H^2}^{1/2}$$

says that

$$\begin{aligned}
\langle \|\nabla\overline{\boldsymbol{u}}\|_\infty \rangle &\leq \left\langle \overline{H_3} \right\rangle^{1/4} \left\langle \overline{H_2} \right\rangle^{1/4} \\
&\leq C(\alpha,\nu,\ell,L)Re^{5/2}.
\end{aligned} \tag{61}$$

(c) Thus we obtain from (56) and (61) that

$$\ell^2\left\langle \kappa_{N,r}^2 \right\rangle \leq C(\alpha,\nu,\ell,L)Re^{5/2} + CRe(\ln Re). \tag{62}$$

In particular, for $r = 0$

$$\ell^2\left\langle \kappa_{N,0}^2 \right\rangle \leq C(\alpha,\nu,\ell,L)Re^{5/2} + CRe(\ln Re). \tag{63}$$

By choosing $r = 1$ we can then get an improvement for $\left\langle \kappa_{N,0}^2 \right\rangle$ by writting

$$\begin{aligned}
\left\langle \kappa_{N,0}^2 \right\rangle &= \left\langle \kappa_{N,1}^{2(N-1)/N}\kappa_{1,0}^{2/N} \right\rangle \\
&\leq \left\langle \kappa_{N,1}^2 \right\rangle^{(N-1)/N} \left\langle \kappa_{1,0}^2 \right\rangle^{1/N},
\end{aligned} \tag{64}$$

and then using the above estimates for $\left\langle \kappa_{N,1}^2 \right\rangle$ and $\left\langle \kappa_{1,0}^2 \right\rangle$ which give for $N \geq 2$,

$$\ell^2\left\langle \kappa_{N,0}^2 \right\rangle \leq C(\alpha,\nu,\ell,L)^{(N-1)/N}Re^{5/2-3/2N}(\ln Re)^{1/N} + CRe\ln Re. \tag{65}$$

Note that when $N = 1$ we return to $\ell^2\left\langle \kappa_{1,0}^2 \right\rangle \leq CRe\ln Re$.

## 5.2 Proof of inequality (11)

Let us come back to relation (3) , when $N = 0$, we get the energy inequality (29)

$$\frac{d}{dt}(\overline{H_0} + \alpha^2 \overline{H_1}) \leq -\nu(\overline{H_1} + \alpha^2 \overline{H_2}) + \overline{H_0}^{1/2} {\Phi_0}^{1/2}. \tag{66}$$

Poincaré inequality together with Cauchy Schwarz, Young and Gronwall inequalities in (66) imply that $\overline{H_0} + \alpha^2 \overline{H_1}$ is uniformly bounded in time according to (25). Time averaging, using the fact that the time average of the time derivative in (66) vanishes, we obtain

$$\begin{aligned} \left\langle \overline{H_1} + \alpha^2 \overline{H_2} \right\rangle &\leq \left\langle \overline{H_0}^{1/2} {\Phi_0}^{1/2} \right\rangle \\ &\leq c\nu^2 \ell^{-4} L^3 Re^3. \end{aligned} \tag{67}$$

Thus

$$\left\langle \overline{H_1} \right\rangle \leq c\nu^2 \ell^{-4} L^3 Re^3. \tag{68}$$

and

$$\left\langle \overline{H_2} \right\rangle \leq c\alpha^{-2} \nu^2 \ell^{-4} L^3 Re^3. \tag{69}$$

From (69) and by using the following interpolation inequality

$$\overline{H_N} \leq \overline{H_{N-s}}^{\frac{r}{r+s}} \overline{H_{N+r}}^{\frac{s}{r+s}}, \tag{70}$$

that is

$$\overline{H_1} \leq \overline{H_0}^{\frac{1}{2}} \overline{H_2}^{\frac{1}{2}}, \tag{71}$$

we can improve (68) in order to obtain

$$\left\langle \overline{H_1} \right\rangle \leq \left\langle \overline{H_0} \right\rangle^{\frac{1}{2}} \left\langle \overline{H_2} \right\rangle^{\frac{1}{2}} \leq c\alpha^{-1} \nu^2 \ell^{-3} L^3 Re^{5/2}. \tag{72}$$

This improve the Navier Stokes result (18) for the inverse Kolomogorov lentgh to

$$\ell \lambda_k^{-1} \leq c(\frac{\ell}{\alpha})^{1/4} Re^{5/8}. \tag{73}$$

We also deduce that the energy dissipation rate $\overline{\epsilon} = \nu \left\langle \|\nabla \overline{\boldsymbol{u}}\|_{L^2}^2 \right\rangle L^{-3}$ is also bounded by $Re^{5/2}$ but all the improved estimates blow up when $\alpha$ tends to zero.

## 5.3 Proof of inequality (12)

The authors in [9] showed the existence of a global attractor $\mathcal{A}$ to this model and they made estimates of the fractal dimension of this attractor. The sharp estimate found in [9] for the fractal dimension of $\mathcal{A}$ expressed in terms of Grashof number $Gr$ is

$$d_{F,ML-\alpha}(\mathcal{A}) \leq c \left( \frac{2\pi}{L\alpha^2 \gamma} \right)^{3/4} Gr^{3/2}. \tag{74}$$

Where

$$\frac{1}{\gamma} = \min{(1, \frac{2\pi}{\alpha^2 L})}$$

This however can be improved by noting that their estimate depends upon $\langle \overline{H_1} + \alpha^2 \overline{H_2} \rangle$ whose upper bound is $Re^3$ not $Gr^2 \leq cRe^4$. With this improvement it is found that the estimate of $d_{F,ML-\alpha}(\mathcal{A})$ in [9] convert to

$$d_{F,ML-\alpha}(\mathcal{A}) \leq c \left( \frac{L^{3/2}(2\pi)^{3/2}\ell^{-4}}{\alpha^2} \right)^{3/4} Re^{9/4}. \tag{75}$$

In term of degrees of fredoom, this result says that $Re^{3/4} \times Re^{3/4} \times Re^{3/4}$ resolution grid points are needed.

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